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THE MOTION OF A SYSTEM CLOSE TO HAMILTONIAN WITH ONE DEGREE OF FREEDOM WHEN THERE IS RESONANCE IN FORCED VIBRATIONS†

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The motion of an autonomous mechanical system with one degree of freedom subject to small time-periodic perturbations and small dissipative forces in the vicinity of a stable position of equilibrium of the system is considered. It is assumed that resonance occurs in forced vibrations when the ratio of the frequency of small vibrations of the system to the frequency of the external periodic perturbation is close to an integer. The qualitative behaviour of an approximate system is studied. Depending on the parameters of the problem, namely, the magnitude of the dissipation and resonance detuning, a rigorous solution of the problem of the existence, number, and stability of periodic motions (the period being equal to that of the perturbation) arising from the position of equilibrium of the unperturbed system is given. As an example the motion of a pendulum with oscillating point of suspension is considered. Copyright © 1996 Elsevier Science Ltd.

1. FORMULATION OF THE PROBLEM. TRANSFORMATION OF THE HAMILTONIAN

Consider a mechanical system close to integrable, the Hamiltonian of which can be represented as a series in powers of a small parameter ε

$$H(x, p_x, t) = H^{(0)}(x, p_x) + \varepsilon H^{(1)}(x, p_x, t) + \varepsilon^2 H^{(2)}(x, p_x, t) + \ldots \qquad (1.1)$$

where $H^{(i)}(x, p_x, t)$ $(i = 1, 2, \ldots)$ are 2π-periodic functions of time.

When $\varepsilon = 0$, let the kinetic and potential energy of the system be $\frac{1}{2}a(x)\dot{x}^2$ and $c(x)$, respectively, the origin $x = 0$ of the system of coordinates being a position of stable equilibrium.

In a neighbourhood of the point $x = 0, p_x = 0$ the functions $H^{(i)}$ can be represented by the series

$$H^{(0)}(x, p_x) = H_2^{(0)} + H_3^{(0)} + H_4^{(0)} + \ldots, \quad H_2^{(0)} = \tfrac{1}{2} p_x^2 / a(0) + \tfrac{1}{2} c''(0) x^2$$
$$H^{(i)}(x, p_x, t) = H_1^{(i)} + H_2^{(i)} + H_3^{(i)} + \ldots, \quad H_1^{(i)} = f_i(t) x + g_i(t) p_x, \quad i = 1, 2, 3, \ldots$$

where $H_k^{(i)}$ is a polynomial of degree k in x and p_x.

We assume that resonance occurs in the system under forced vibrations, that is, the frequency $\omega_0 = \sqrt{(c''(0)/a(0))}$ of small vibrations of the system is close to an integer when $\varepsilon = 0$. In addition, we shall assume that the system is subject to dissipative forces described by Rayleigh's function of the form $R(\dot{x}) = \frac{1}{2}\delta\dot{x}^2$.

The purpose of this paper is to study the problem of the existence, number and stability of 2π-periodic motions of the system depending on the parameters of the problem, namely, the magnitude of the dissipation and the closeness of the frequency ω_0 to an integer. Moreover, we shall study the qualitative behaviour of an approximate (model) system in the vicinity of a position of equilibrium ($\varepsilon = 0$) of the unperturbed system in the resonance case considered.

First, we will introduce a number of canonical replacements of variables, which simplify the structure of the Hamiltonian (1.1). Setting

$$x = \varepsilon^{1/3} x^* / \sqrt{\omega_0 a(0)}, \quad p_x = \varepsilon^{1/3} \sqrt{\omega_0 a(0)}\, p_x^*$$

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we write the new Hamiltonian in the form

$$
\begin{aligned}
&H^*(x^*, p_x^*, t) = \tfrac{1}{2}\omega_0(x^{*2} + p_x^{*2}) + \varepsilon^{1/3} H_3^{(0)}(x^*, p_x^*) + \\
&+\varepsilon^{2/3} H_4^{(0)}(x^*, p_x^*) + \varepsilon^{2/3} f(t) x^* + \varepsilon^{2/3} g(t) p_x^* + O(\varepsilon) \\
&f(t) = f_1(t) / \sqrt{\omega_0 a(0)}, \quad g(t) = g_1(t)\sqrt{\omega_0 a(0)}
\end{aligned}
\tag{1.2}
$$

The corresponding equations of motion can be written as follows:

$$
dx^* / dt = \partial H^* / \partial p_x^*, \quad dp_x^* / dt = -\partial H^* / \partial x^* - \varepsilon^{2/3}\delta_* p_x^* + O(\varepsilon)
$$

The quantity δ_* is defined by $\delta = \varepsilon^{2/3} a(0)\delta_*$.

Henceforth, carrying out a canonical transformation x^*, $p_x^* \to \xi$, η of the Birkhoff type, we shall make the form $H_3^{(0)}$ in (1.2) vanish and we shall simplify the form $H_4^{(0)}$, so that the Hamiltonian becomes

$$
K = \tfrac{1}{2}\omega_0(\xi^2 + \eta^2) + \tfrac{1}{4}\varepsilon^{2/3} c_2 (\xi^2 + \eta^2)^2 + \varepsilon^{2/3} f(t)\xi + \varepsilon^{2/3} g(t)\eta + O(\varepsilon) \tag{1.3}
$$

and the equations of motion will be

$$
d\xi / dt = \partial K / \partial \eta, \quad d\eta / dt = -\partial K / \partial \xi - \varepsilon^{2/3}\delta_* \eta + O(\varepsilon) \tag{1.4}
$$

We assume that the constant coefficient c_2 in (1.3) is non-zero.

Let $\omega_0 = N + \varepsilon^{2/3}\mu_*$, where N is an integer. We represent $f(t)$ and $g(t)$ as Fourier series

$$
\begin{aligned}
f(t) &= a_N \cos Nt + b_N \sin Nt + \sum_{n \neq N} (a_n \cos nt + b_n \sin nt) \\
g(t) &= c_N \cos Nt + d_N \sin Nt + \sum_{n \neq N} (c_n \cos nt + d_n \sin nt)
\end{aligned}
$$

Discarding the terms $O(\varepsilon)$ and setting $\delta_* = 0$ in (1.4), we shall consider linear equations of motion. Their solutions of the form

$$
\begin{aligned}
\xi^* &= \varepsilon^{2/3} \sum_{n \neq N} [(-\omega_0 a_n + n d_n)\cos nt - (\omega_0 b_n + n c_n)\sin nt] / (\omega_0^2 - n^2) \\
\eta^* &= \varepsilon^{2/3} \sum_{n \neq N} [-(\omega_0 c_n + n b_n)\cos nt + (n a_n - \omega_0 d_n)\sin nt] / (\omega_0^2 - n^2)
\end{aligned}
$$

represent forced oscillations of the system corresponding to non-resonance external perturbation frequencies when there is no dissipation. Setting

$$
\xi_1 = \xi - \xi^*, \quad \eta_1 = \eta - \eta^*
$$

we make the terms with non-resonance frequencies vanish in the terms $\varepsilon^{2/3} f(t)\xi$ and $\varepsilon^{2/3} g(t)\eta$ of the Hamiltonian (1.3).

Then, changing to new canonical variables φ, r by $\xi_1 = \sqrt{(2r)} \sin \varphi$, $\eta_1 = \sqrt{(2r)} \cos \varphi$ and equating to zero the terms containing the harmonics $\sin(\varphi + Nt)$, $\cos(\varphi + Nt)$ in the new Hamiltonian, by means of the substitution $\varphi, r \to \varphi_*, r_*$ given by

$$
r = r_* + \varepsilon^{2/3}\sqrt{2r_*}[(b_N - c_N)\cos(\varphi_* + Nt) - (a_N + d_N)\sin(\varphi_* + Nt)] / (4N) + O(\varepsilon)
$$

$$
\varphi = \varphi_* - \sqrt{2}\varepsilon^{2/3}[(b_N - c_N)\sin(\varphi_* + Nt) + (a_N + d_N)\cos(\varphi_* + Nt)] / (8N\sqrt{r_*}) + O(\varepsilon)
$$

we obtain a Hamiltonian of the form

$$\Gamma = Nr_* + \varepsilon^{2/3}\mu_* r_* + \varepsilon^{2/3} c_2 r_*^2 + \varepsilon^{2/3}\sqrt{r_*}\,\varkappa \cos(\varphi_* - Nt - \gamma) + O(\varepsilon) \tag{1.5}$$

where the angle γ is defined by the relations $\sin\gamma = (a_N - d_N)/\varkappa_1$, $\cos\gamma = (b_N + c_N)\,\varkappa_1$ and $\varkappa_1 = \sqrt{(a_N - d_N)^2 + (b_N + c_N)^2}$, while $\varkappa = \varkappa_1/\sqrt{2}$.

The equations of motion will be

$$d\varphi_* / dt = \partial\Gamma / \partial r_* + \varepsilon^{2/3}\delta_* \sin\varphi_* \cos\varphi_* + O(\varepsilon)$$
$$dr_* / dt = -\partial\Gamma / \partial\varphi_* - \varepsilon^{2/3}\delta_* r_* (1 + \cos 2\varphi_*) + O(\varepsilon) \tag{1.6}$$

A replacement of variables $\varphi_*, r_* \to \psi, R$ of the form

$$\varphi_* = \psi - \varepsilon^{2/3}\delta_* \cos 2\psi / (4N) + O(\varepsilon)$$
$$r_* = R - \varepsilon^{2/3}\delta_* R \sin 2\psi / (2N) + O(\varepsilon)$$

enables us to simplify the dissipative terms in (1.6). Then the new Hamiltonian will have the form (1.5), with ψ and R in place of φ_* and r_*, while the equations become

$$d\psi / dt = \partial\Gamma / \partial R + O(\varepsilon), \quad dR / dt = -\partial\Gamma / d\psi - \varepsilon^{2/3}\delta_* R + O(\varepsilon)$$

Finally, the change of variables

$$\psi = Nt + \gamma + \sigma(\theta + \pi/2) - \pi/2, \quad R = (\varkappa / c_2)^{2/3}\rho \quad (\sigma = \operatorname{sign} c_2)$$

and the introduction of a new time τ by the formula $\tau = \varepsilon^{2/3} |c_2|^{1/3} \varkappa^{2/3} t$ reduce the Hamiltonian (1.5) to the form

$$H = -\mu\rho + \rho^2 + \sqrt{\rho}\cos\theta + O(\varepsilon^{1/3}), \quad \mu = -\sigma\mu_* / (|c_2|^{1/3}\varkappa^{2/3}) \tag{1.7}$$

The corresponding equations of motion can be written as follows:

$$d\theta / d\tau = \partial H / \partial\rho + O(\varepsilon^{1/3}), \quad d\rho / d\tau = -\partial H / \partial\theta - \chi\rho + O(\varepsilon^{1/3}) \tag{1.8}$$
$$\chi = \delta_* / (|c_2|^{1/3}\varkappa^{2/3})$$

2. INVESTIGATION OF A MODEL SYSTEM

Discarding the terms $O(\varepsilon^{1/3})$ in (1.7) and (1.8), we obtain a truncated (model) system. Its motion can be described by the equations

$$d\theta/d\tau = -\mu + 2\rho + \cos\theta/(2\sqrt{\rho}), \quad d\rho/d\tau = \sqrt{\rho}\sin\theta - \chi\rho \tag{2.1}$$

We shall study the qualitative behaviour of (2.1) for various values of χ and μ.

The positions of equilibrium $\theta = \theta_*$, $\rho = \rho_*$ of system (2.1) satisfy the equations

$$\sin\theta = \chi\sqrt{\rho}, \quad \cos\theta = 2\sqrt{\rho}\,(\mu - 2\rho)$$

whence, eliminating θ, we obtain the following equation for ρ

$$F(\rho) \equiv \rho^3 - \mu\rho^2 + \tfrac{1}{16}(\chi^2 + 4\mu^2)\rho - \tfrac{1}{16} = 0 \tag{2.2}$$

The number of real roots of (2.2) depends on the sign of the expression

$$Q(\chi,\mu) = \chi^6 + 8\chi^4\mu^2 + 16\chi^2\mu^4 - 8\mu(4\mu^2 + 9\chi^2) + 108 \tag{2.3}$$

If $Q > 0$, Eq. (2.2) has one real root. If $Q = 0$, it has three roots, at least two of which are equal. And if $Q < 0$, there are three different real roots. Analysing $F(\rho)$, we can see that all real roots of the equation $F(\rho) = 0$ are positive.

The curve $Q(\chi, \mu) = 0$, drawn by a computer, is shown in Fig. 1. The curve $Q = 0$ has a horizontal tangent at (0, 3/2) and it approaches the $O\mu$ axis asymptotically as $\mu \to +\infty$, the point $P(3^{1/2}2^{-1/3}, 3 \cdot 2^{4/3})$ being a cuspidal point.

System (2.1) has one position of equilibrium in domain I ($Q > 0$) and three in domain III ($Q < 0$).

To study the stability of the above positions of equilibrium we set $\theta = \theta_* + x$, $\rho = \rho_* + y$. Then, from (2.1) we obtain a linearized system of equations, the roots of its characteristic equation having the form

$$\lambda_{1,2} = -\tfrac{1}{2}\chi \pm \sqrt{(6\rho_* - \mu)(\mu - 2\rho_*)} \tag{2.4}$$

When

$$s(\chi,\mu) \equiv \chi^2/4 - (6\rho_* - \mu)(\mu - 2\rho_*) > 0 \tag{2.5}$$

both roots (2.4) have negative real parts and the position of equilibrium under consideration is asymptotically stable. If $s(\chi, \mu) < 0$, one of the roots (2.4) is positive, the position of equilibrium being unstable.

From (2.2) and (2.5) we obtain

$$s(\chi, \mu) = 4F'(\rho_*)$$

i.e. the sign of $s(\chi, \mu)$ is the same as that of the derivative $F'(\rho)$ at the equilibrium point under consideration (a zero of $F(\rho)$). Therefore, the corresponding position of equilibrium is asymptotically stable if $F(\rho)$ increases on passing through zero, and unstable if it decreases.

Hence, analysing $F(\rho)$, we find that in domain I the unique position of equilibrium of system (2.1) is asymptotically stable; in domain III the upper and lower roots of (2.2) correspond to asymptotically stable positions of equilibrium, while the middle root corresponds to an unstable position of equilibrium.

An unstable position of equilibrium corresponds to a saddle point in the phase plane of (2.1) and a stable position of equilibrium corresponds either to a stable focus when $(6\rho_* - \mu)(\mu - 2\rho_*) < 0$) or to a stable node (when the converse inequality is satisfied).

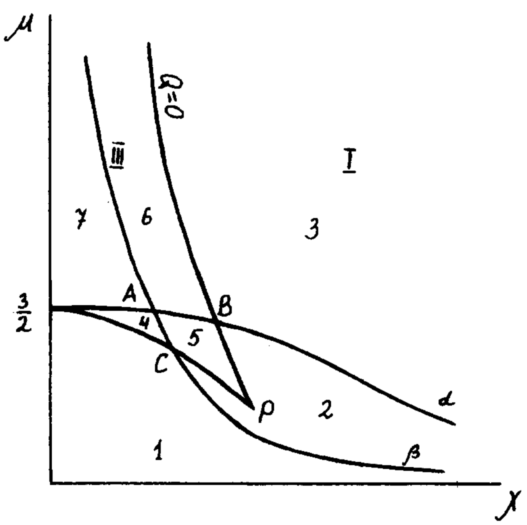

Fig. 1.

The type of singular points corresponding to asymptotically stable positions of equilibrium changes as the curves (in the χ, μ-plane) corresponding to $\rho_* = \mu/6$ and $\rho_* = \mu/2$ are crossed. Setting $F(\mu/6) = 0$ and $F(\mu/2) = 0$, from (2.2) we obtain the following equations for the curves (in Fig. 1 they are denoted by α and β, respectively)

$$\chi^2 = (54 - 16\mu^3)/(9\mu), \quad \mu = 2/\chi^2 \tag{2.6}$$

The points A, B, and C of intersection of these curves and the curve $Q = 0$ have coordinates $(2^{5/6}3^{1/3}, (3/2)^{2/3})$, $(3 \cdot 2^{1/6} \cdot 5^{-5/6}, 3 \cdot (2/25)^{1/3}$ and $(2^{1/3}, 2^{1/3})$, respectively. At the point $(0, 3/2)$ curve α has a common horizontal tangent with the curve $Q = 0$, and it approaches the O_χ axis asymptotically as $\chi \to +\infty$.

The curves α, β and $Q = 0$ divide the half-plane $\chi > 0$ in the χ, μ plane into seven subdomains (indicated by numbers 1–7 in Fig. 1), in which the trajectories of (2.1) behave differently. The corresponding phase space patterns are presented in Fig. 2(a)–(g) in the plane of $u = \sqrt{(2\rho)} \cos\theta$, $\upsilon = \sqrt{(2\rho)}\sin\theta$.

A stable focus in subdomains 1 and 3 of domain I (Fig. 2a and c) and a stable node in subdomain 2 (Fig. 2b) correspond to an asymptotically stable position of equilibrium.

In domain III we distinguish four subdomains 4–7. The lower root of (2.2) corresponds to a stable node in subdomains 4, 5 and a stable focus in subdomains 6, 7. The upper root corresponds to a stable focus in subdomains 4, 7 and a stable node in subdomains 5, 6. A saddle point corresponds to the middle root. The phase space patterns of (2.1) corresponding to subdomains 4–7 are presented in Fig. 2(d)–(g).

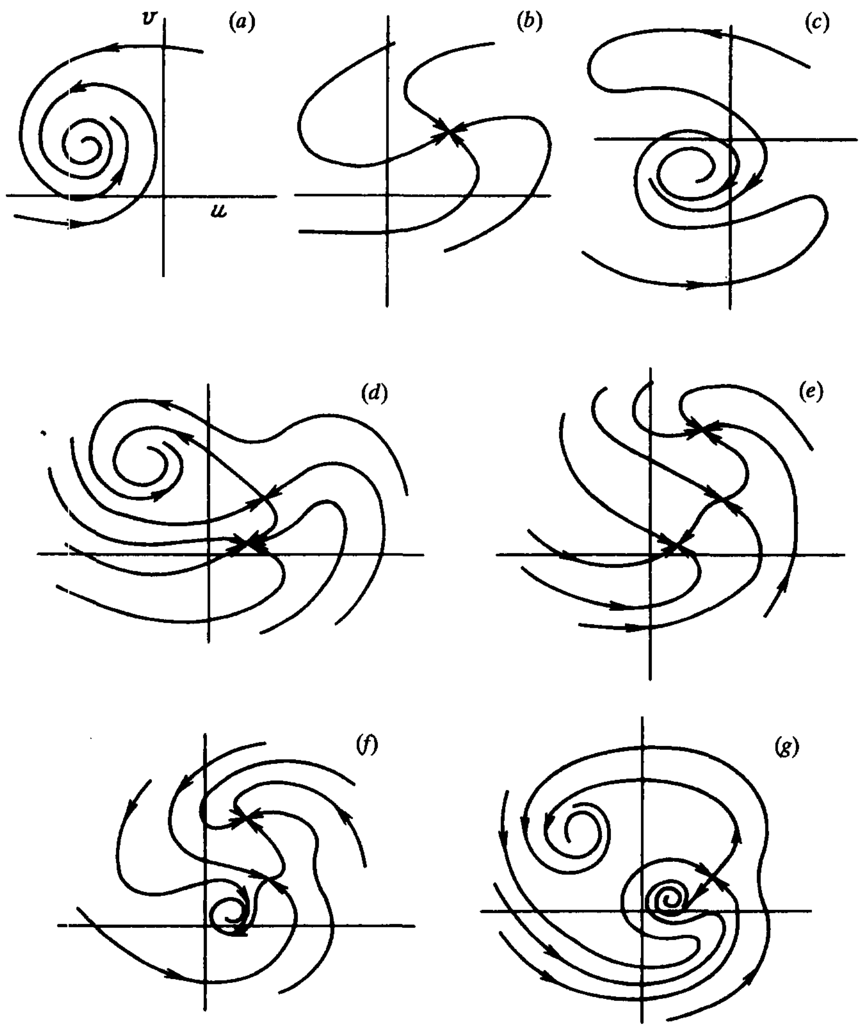

Fig. 2.

We observe that when $\chi > 0$, (2.1) has no closed trajectories, which follows from the Bendixon criterion [1] and the form of the right-hand sides of (2.1). Each trajectory of the system converges to one (in domain I) or one of two (in domain III) asymptotically stable positions of equilibrium.

For the curves α, β, and $Q = 0$ the roots of Eq. (2.2) can be given explicitly:

for the points of curve α

$$\rho_1 = \mu/6, \quad \rho_{2,3} = \tfrac{5}{12}\mu \mp \sqrt{25\mu^3 - 54}/(12\sqrt{\mu})$$

(in domain I the roots $\rho_{2,3}$ are complex conjugate)

for the points of curve β

$$\rho_{1,2} = \mu/4 \mp \tfrac{1}{4}\sqrt{(\mu^3 - 2)/\mu}, \quad \rho_3 = \mu/2$$

(in domain I the roots $\rho_{1,2}$ are complex conjugate)

for the points of the lower branch of the curve $Q = 0$

$$\rho_1 = \rho_2 = (4\mu - \sqrt{4\mu^2 - 3\chi^2})/12, \quad \rho_3 = (16\rho_1^2)^{-1}$$

for the points of the upper branch of $Q = 0$

$$\rho_1 = (16\rho_2^2)^{-1}, \quad \rho_2 = \rho_3 = (4\mu + \sqrt{4\mu^2 - 3\chi^2})/12$$

for the point P on the curve $Q = 0$

$$\rho_1 = \rho_2 = \rho_3 = 2^{-4/3}$$

We will describe the changes affecting the singular points on these curves without presenting the phase space patterns of (2.1) corresponding to the curves α, β and $Q = 0$.

On curves α and β the singular point corresponding, respectively, to the equilibrium values $\rho_1 = \mu/6$ and $\rho_3 = \mu/2$ changes character (from a stable focus to a stable node or vice versa). For these points the characteristic equation has a multiple root. They correspond to a stable degenerate node in the phase plane. The character of the remaining singular points (in domain III) on α and β remains unchanged.

On the curve $Q = 0$ two singular points from domain III, a saddle and a stable node, are combined into one compound singular point of saddle-node type with stable node sector [1] (the corresponding characteristic equation has one zero root and one negative real root). In domain I this compound singular point disappears and a single singular point remains, the type of which remains unchanged as the curve $Q = 0$ is crossed.

At the point P of the curve $Q = 0$ the three singular points of (2.1) are combined into one compound singular point, which is a stable node [1].

3. PERIODIC SOLUTIONS OF THE COMPLETE SYSTEM

We consider the problems of the existence and stability of periodic solutions of the complete system of equations (1.8) with Hamiltonian (1.7) arising from the positions of equilibrium of the model system (2.1) for the values of χ and μ inside domains 1–7 in Fig. 1. In the neighbourhood of the position of equilibrium $\theta = \theta_*$, $\rho = \rho_*$, (1.8) can be regarded as a quasi-linear system with perturbations of order $\varepsilon^{1/3}$ having period $T_* \sim \varepsilon^{2/3}$ in τ. Because the roots of the characteristic equation of the model system are of order one, they cannot be equal to $ik2\pi/T$ (k is an integer), what occurs is the non-resonance case of Poincaré's theory in the problem of periodic notions of quasilinear systems [2].

Each position of equilibrium of the model system gives rise to one solution of (1.8) which is T-periodic in τ and analytic in $\varepsilon^{1/3}$. In the original system, which is close to being Hamiltonian, this corresponds to a solution which is analytic in $\varepsilon^{1/3}$ and periodic in time with period equal to that of the external perturbation. When $\varepsilon = 0$ this periodic solution corresponds to the stable position of equilibrium $x \equiv 0$ of the unperturbed system.

The conclusions concerning the stability of the positions of equilibrium of the model system can be extended to the corresponding periodic solutions of the complete system: the asymptotically stable and unstable positions of equilibrium of (2.1) turn into asymptotically stable and, respectively, unstable periodic solutions of (1.8). This follows from the continuity with respect to ε of the characteristic exponents of the corresponding linear equations of perturbed motion.

4. EXAMPLE: A PENDULUM WITH AN OSCILLATING POINT OF SUSPENSION

As an example consider the motion of a mathematical pendulum whose point of suspension undergoes horizontal harmonic oscillations of small amplitude. Let l be the length of the pendulum, let x be the angle between the pendulum and the vertical direction, and let a and Ω be the amplitude and frequency of oscillations and the point of suspension. The pendulum is subject to dissipative forces given by the Rayleigh formula $R = \frac{1}{2}\delta x'^2$ (here and henceforth a prime denotes differentiation with respect to the dimensionless "time" Ωt, which will be denoted by t below).

The motion of the pendulum can be described by the equation

$$x'' + \delta x' + \omega_0^2 \sin x = \varepsilon \sin t \cos x \tag{4.1}$$

$$\varepsilon = a/l \ll 1, \quad \omega_0^2 = g/(\Omega^2 l)$$

This equation can be replaced by an equivalent system of two equations of the form

$$x' = \partial H/\partial p_x, \quad p_x' = -\partial H/\partial x - \delta x' \tag{4.2}$$

$$H = \tfrac{1}{2}p_x^2 - \omega_0^2 \cos x - \varepsilon \sin t \sin x$$

When $\varepsilon = 0$ and $\delta = 0$ system (4.2) has the solution $x = 0, p_x = 0$, which corresponds to a stable position of equilibrium of the pendulum. In a neighbourhood of this solution the Hamiltonian can be represented by the series

$$H = \tfrac{1}{2}(p_x^2 + \omega_0^2 x^2) - \tfrac{1}{24}\omega_0^2 x^4 + \varepsilon(-\sin t \cdot x + \tfrac{1}{6}\sin t \cdot x^3) + \dots \tag{4.3}$$

where the dots stand for terms of degree higher than four in x and p_x.

Suppose that the frequency ω_0 of small characteristic oscillations of the pendulum is close to one. We shall assume that $\omega_0 = 1 + 2^{-5/3}\varepsilon^{2/3}\mu$ and also $\delta = \varepsilon^{2/3}\delta_*$.

Following the discussion in Section 1, we perform a number of canonical coordinate transformations which reduce the Hamiltonian (4.3) to the form (1.7) and the system of equations of motion to the form (1.8). This sequence of coordinate transformations has the form

$$x = \varepsilon^{1/3}x^*/\sqrt{\omega_0}, \quad p_x = \varepsilon^{1/3}\sqrt{\omega_0}\,p_x^* \tag{4.4}$$

$$x^* = \xi + \varepsilon^{2/3}(5\xi^3 + 9\xi\eta^2)/(192\omega_0) + O(\varepsilon^{4/3})$$

$$p_x^* = \eta - \varepsilon^{2/3}(5\xi^2\eta + \eta^3)/(64\omega_0) + O(\varepsilon^{4/3}) \tag{4.5}$$

$$\xi = \sqrt{2r}\sin\varphi, \quad \eta = \sqrt{2r}\cos\varphi \tag{4.6}$$

$$\varphi = \varphi_* + \sqrt{2}\varepsilon^{2/3}\sin(\varphi_* + t)/(8\sqrt{r_*}) + O(\varepsilon^{4/3}) \tag{4.7}$$

$$r = r_* - \tfrac{1}{4}\varepsilon^{2/3}\sqrt{2r_*}\cos(\varphi_* + t) + O(\varepsilon^{4/3})$$

$$\varphi_* = \psi - \tfrac{1}{4}\varepsilon^{2/3}\delta_*\cos 2\psi + O(\varepsilon^{4/3}) \tag{4.8}$$

$$r_* = R - \tfrac{1}{2}\varepsilon^{2/3}\delta_* R\sin 2\psi + O(\varepsilon^{4/3})$$

$$\psi = t - \theta, \quad R = 2^{7/3}\rho \tag{4.9}$$

Along with the last transformation, we introduce the new independent variable $\tau = 2^{-5/3}\varepsilon^{2/3}t$. As a result of the transformations (4.4)–(4.9), the system of equations (4.2) takes the form

$$d\theta/d\tau = \partial H/\partial\rho + O(\varepsilon^{2/3}), \quad d\rho/d\tau = -\partial H/\partial\theta - \chi\rho + O(\varepsilon^{2/3})$$

$$H = -\mu\rho + \rho^2 + \sqrt{\rho}\cos\theta + O(\varepsilon^{2/3}), \quad \chi = 2\delta_*$$

From the results of Sections 2 and 3 it follows that the stable position of equilibrium of the pendulum which exists for $\varepsilon = 0$ gives rise to one or three 2π-periodic motions of the pendulum when $0 < \varepsilon \ll 1$ (respectively, in domains I and III in Fig. 1), to which there correspond one or three positions of equilibrium of the model system (Section 2). Determining the equilibrium values of θ and ρ (we denote them by θ_0 and ρ_0 for domain *I* and by θ_i, ρ_i ($i = 1, 2, 3$) for domain III) and applying the transformations (4.4)–(4.9) in the reverse order, we find that in domain I the 2π-periodic solution has the form

$$x_0(t) = 2^{5/3}\varepsilon^{1/3}\sqrt{\rho_0}\sin(t-\theta_0) + O(\varepsilon^{2/3}) \tag{4.10}$$

and in domain III

$$x_i(t) = 2^{5/3}\varepsilon^{1/3}\sqrt{\rho_i}\sin(t-\theta_i) + O(\varepsilon^{2/3}), \quad i = 1,2,3 \tag{4.11}$$

The solutions (4.10) and (4.11) describe small-amplitude oscillations of the pendulum with frequency equal to that of the point of suspension. The initial angle between the pendulum and the vertical direction for the oscillations in question is negative (the pendulum is displaced to the left). The initial angular velocity $x_0'(0) < 0$ in subdomain 1 and $x_0'(0) > 0$ in subdomains 2 and 3 (see Fig. 1); $x_2'(0) > 0, x_3'(0) > 0$ in domain III, $x_1'(0) > 0$ in subdomains 5, 6, and $x_1'(0) < 0$ in subdomains 4, 7.

From the results of Section 3 it follows that in domain I the unique periodic solution $x_0(t)$ is asymptotically stable. In domain III the solutions $x_1(t)$ and $x_3(t)$ corresponding to the oscillations of the pendulum having the smallest and, respectively, the largest amplitude are also asymptotically stable, while the solution $x_2(t)$ corresponding to oscillations with the middle amplitude is unstable.

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